Lecture 3: Vertical areas, calculating volumes using integrals, washer method, volumes of rotational objects

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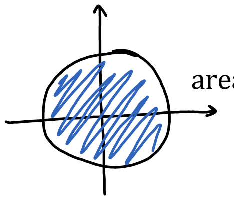
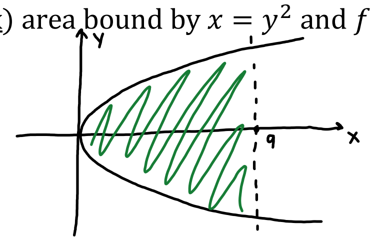
Vertical area: Sometimes we're interested in the area enclosed by a curve that is not a function.

Ex)



area of a circle

Ex) area bound by $x = y^2$ and $f(y) = 9$



1) we could computed the inverse function of $x = y^2$ and proceed as before, but sometimes this is impossible.
2) INSTEAD, we're going to integrate over y. (this is simpler (ie. less room for error), and often faster)

Step 1: Determine the bounds of the integral.

$y^2 = 9$ set 2 curves equal

$y = \pm 3$ solve for y

Step 2: Determine the boundary further right (before, we needed to know which curve was above the other).

Here, $f(y) = 9$ is further to the right.

So:

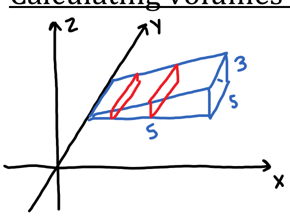
$$A = \int_{-3}^{3} (9 - y^2)\, dy$$

$$= 9y - \frac{1}{3} \cdot y^3 \Big]_{-3}^{3}$$

$$= 9(3) - \frac{1}{3}(3)^2 - 9(-3) + \frac{1}{3}(-3)^3$$

$$= 36 \text{ units}^2$$

Calculating volumes using integrals



this is a wedge (half of a box)

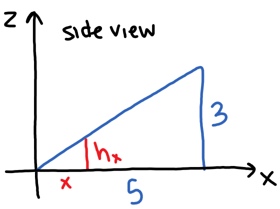
if we wanted to find the volume of this wedge,

we'd use our elementary formula $V_{\frac{1}{2}box} = \frac{lwh}{2} = \frac{5(5)(3)}{2} = \frac{75}{2}$

Alternatively (specifically once we deal with shape without a nice volume formula), we can use integrals.

To use integrals, we take the cross-sectional area of the shape.

Notice above that the rectangle of the cross-section can have a different height throughout the shape. This means we need a function in x that describes the cross-section area as it changes.

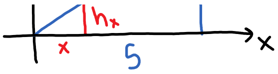


use similar triangles to determine h_x in terms of x.

$\frac{h_x}{x} = \frac{3}{5}$, $\quad h_x = \frac{3}{5}x$

ie. the area of the cross section at any point x is $\frac{3}{5}x$

and the **volume** of the shape is the integral over the **cross-section area**

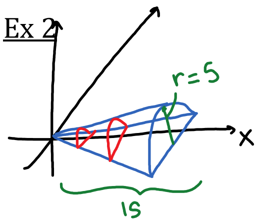


ie. the area of the cross section at any point x is $\frac{3}{5}x$
and the **volume** of the shape is the integral over the **cross-section area** function.

length of wedge (constant)

$$\int_0^5 5\cdot\frac{3}{5}x\,dx$$
$$=\frac{3}{2}x^2\Big|_0^5$$
$$=\frac{3}{2}(25)$$
$$=\frac{75}{2}$$

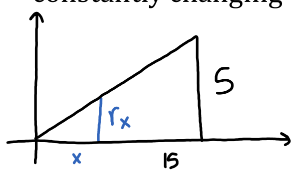
Ex 2



(1) cross sections are half circles with radius r=5

$$A=\frac{\pi r^2}{2}$$

(2) determine radius depending on x since r is constantly changing



$$\frac{r_x}{x}=\frac{5}{15}=\frac{1}{3},\qquad r_x=\frac{x}{3}$$

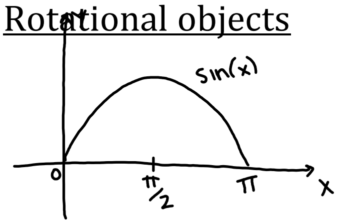
(3) cross section **area** of half-circle at x

$$A(x)=\left(\frac{x}{3}\right)^2\cdot\frac{\pi}{2}$$
$$=\frac{\pi x^2}{18}$$

(4) integrate with respect to x to compute volume

$$V=\int_0^{15}\frac{\pi x^2}{18}\,dx$$
$$=\frac{\pi}{18}\int_0^{15}x^2\,dx$$
$$=\frac{\pi}{18}\left(\frac{x^3}{3}\Big|_0^{15}\right)$$
$$=\frac{125}{2}\pi$$

Rotational objects



If we rotate a function, the cross-section area of the object is always a circle. The radius at x is the function value at x, and the cross-section area function is:

$$A(x)=\left(f(x)\right)^2\cdot\pi$$

Therefore, the volume is:

$$\int_0^{\pi}\left(f(x)\right)^2\cdot\pi\,dx$$

In the case of $f(x)=\sin(x)$:

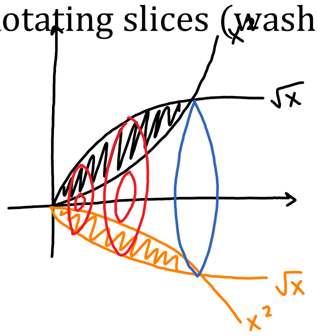
$$\int_0^{\pi}\sin^2 x\cdot\pi\;dx$$
$$=\pi\int_0^{\pi}\frac{1}{2}(1-\cos(2x))\;dx$$
$$=\pi\left(x-\frac{\sin(2x)}{2}\right)\Big]_0^{\pi}$$
$$\pi^2$$

Half angle formula

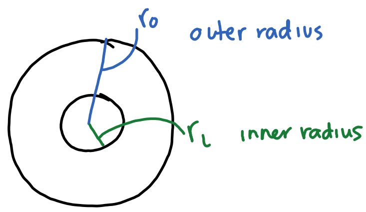
$\sin^2 x=\frac{1}{2}(1-\cos(2x))$

$$= \pi\left(x - \frac{\sin(2x)}{2}\right)\Bigg]_0$$
$$= \frac{\pi^2}{2}$$

Rotating slices (washer method)



On the left, we have the area between $y = x^2$ and $y = \sqrt{x}$ rotated around the x-axis. The cross sections of this shape are annule:



here: $r_i(x)$ is the value of x^2 at x
$r_o(x)$ is the value of $\sqrt{x}$ at x

to determine which is inner or outer (if you can't visualize or sketch) you need to find out which function is larger and the intersection points

Integration bounds:
$x^2 = \sqrt{x}, \qquad x = 0, x = 1$

Larger function:
check values @ $x = \frac{1}{2}$: $\left(\frac{1}{2}\right)^2 < \sqrt{\frac{1}{2}}$, so $\sqrt{x}$ is bigger

Area of annulus with radi r_i, r_o:
$A = \underbrace{r_o^2\pi}_{\text{area of outer function}} - \underbrace{r_i^2\pi}_{\text{area of inner function}} = \pi\left(r_o^2 - r_i^2\right)$ radi are function values

Since the radi are function values, the cross-section area in our example is:
$A(x) = \pi\left(\big(\underbrace{\sqrt{x}}_{\text{bigger function}}\big)^2 - \big(\underbrace{x^2}_{\text{smaller function}}\big)^2\right) = \pi(x - x^4)$

So the volume is:
$$V = \int_0^1 \pi(x - x^4)\,dx = \pi\int_0^1 (x - x^4)\,dx$$
$$= \pi\left(\frac{1}{2} - \frac{1}{5}\right)$$
$$= \frac{3\pi}{10}$$